# Eigenspaces in Recursive Sequences*

Ben Galin†

September 5, 2005

One of the areas of study in discrete mathematics deals with sequences, in particular, infinite sequences. An infinite sequence can be defined[1] as a function $f : \mathbb{Z}^* \to \mathbb{R}$ that maps a given nonnegative integer ($\mathbb{Z}^*$) to a corresponding value. The nonnegative integers can be thought of, then, as the position indices of their associated values. More commonly, however, we use the notation $a_k$ rather than $f(k)$ to denote the $k$th element in a sequence.

Occasionally, we are given a recurrence relation between an element in a sequence, $a_k$, and its preceding ones, $a_{k-1}, a_{k-2}, \ldots, a_{k-i}$, where $i$ is a positive integer and $i < k$. In such a case, computing the value of $a_k$ for large values of $k$ may be tedious, if not impractical, and it is of interest to derive the relation between $a_k$ and $k$ directly.

A special case of a recurrence relation in a sequence is the *second-order linear homogeneous recurrence relation with constant coefficients.* This, by definition, is the set of all recurrence sequences of the form

$$a_k = A \cdot a_{k-1} + B \cdot a_{k-2} \;, \tag{1}$$

for all integers $k \geq 2$ and real numbers $A$ and $B$ with $B \neq 0$.

It should be noted that the set of second-order linear homogenous recurrence relation is bigger than it may seem at first. For instance, consider the sequence $1, 2, 4, 8, \ldots, 2^k, \ldots$, for all intergers $k \geq 0$. This sequence may be defined recursively as $a_k = 2a_{k-1}$ and $a_0 = 1$. However, it can also be defined as a second-order recurrence relation: $a_k = a_{k-1} + 2a_{k-2}$, with $a_0 = 1$ and $a_1 = 2$.

In the following discussion, we first examine how relation (1) can be manipulated so that there is a direct relation between $a_k$ and $k$. Next, we provide a method of finding the sequences that satisfy this relation. We will have to consider different cases, as the way in which the sequences are constructed depends on the values of $A$ and $B$. Last, we focus on two specific recursive relations and analyze their behavior in greater depth; the first sequence we examine is the *Fibonacci sequence* and the second is the family of *arithmetic sequences*. Selected topics that require more rigorous examination are discussed in the appendix.



†I would especially like to thank Rick Taylor; much of the following is a derivation of his work.

[1] A more rigorous definition of a sequence is given in the appendix.

# 1 Finding the Characteristic Equation

We begin by rewriting relation (1) as a system of two linear equations:

$$\begin{cases} a_k &= A \cdot a_{k-1} + B \cdot a_{k-2} \\ a_{k-1} &= 1 \cdot a_{k-1} + 0 \cdot a_{k-2} \end{cases} .$$

In a matrix form, this translates to

$$\begin{bmatrix} a_k \\ a_{k-1} \end{bmatrix} = \begin{bmatrix} A & B \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_{k-1} \\ a_{k-2} \end{bmatrix} .$$

Similarly, if $k \geq 3$, the recurrence relation can be applied again to yield

$$\begin{bmatrix} a_k \\ a_{k-1} \end{bmatrix} = \begin{bmatrix} A & B \\ 1 & 0 \end{bmatrix}^2 \begin{bmatrix} a_{k-2} \\ a_{k-3} \end{bmatrix} .$$

We can repeat this process until reaching the initial values $a_0$ and $a_1$, resulting in

$$\begin{bmatrix} a_k \\ a_{k-1} \end{bmatrix} = \begin{bmatrix} A & B \\ 1 & 0 \end{bmatrix}^{k-1} \begin{bmatrix} a_1 \\ a_0 \end{bmatrix} . \tag{2}$$

From linear algebra we know that if a matrix $W$ is diagonalizable, then $W^k$ can be rewritten as $PU^kP^{-1}$, where $U$ is a diagonal matrix with the eigenvalues of $W$ as its entries and $P$ is a matrix of column eigenvectors. The $i$th eigenvector in $P$ corresponds to the $i$th eigenvalue in $U$. Furthermore, $W$ is guaranteed to be diagnolizable if it has distinct eigenvalues. We will examine the case of indistinct eigenvalues later, but for now we assume that this restriction is satisfied.

Recall that $\lambda$ is an eigenvalue of a matrix $W$ iff it satisfies the equation

$$\det(W - \lambda I) = 0,$$

where $I$ is the identity matrix.

Let $W$ be the *transition matrix*. That is, $W = \left[\begin{smallmatrix} A & B \\ 1 & 0 \end{smallmatrix}\right]$. We proceed by finding the eigenvalues of $W$.

$$\begin{aligned} \det\left(\begin{bmatrix} A & B \\ 1 & 0 \end{bmatrix} - \lambda I\right) &= 0 \\ \det\begin{bmatrix} A - \lambda & B \\ 1 & -\lambda \end{bmatrix} &= 0 \\ (A - \lambda)(-\lambda) - B &= 0 \\ \lambda^2 - A\lambda - B &= 0 \,. \end{aligned} \tag{3}$$

We call equation (3) the *characteristic equation* of matrix $W$. The characteristic equation can have two distinct real roots, a repeated real root, or a pair of complex conjugate roots. For now, we are only interested in the first case. We let the two roots be $\lambda_1, \lambda_2$.

### 1.1 Two Distinct Real Eigenvalues

Next, we find two linearly independent eigenvectors of $W$. Let $\mathbf{p_1} = \left[\begin{smallmatrix} p_{1,1} \\ p_{1,2} \end{smallmatrix}\right]$ be an eigenvector associated with $\lambda_1$. Then it must be the case that

$$\begin{aligned} \begin{bmatrix} A & B \\ 1 & 0 \end{bmatrix} \mathbf{p_1} &= \lambda_1 \mathbf{p_1} \\ \begin{bmatrix} A - \lambda_1 & B \\ 1 & -\lambda_1 \end{bmatrix} \mathbf{p_1} &= \mathbf{0} \\ 1 \cdot p_{1,1} - \lambda_1 \cdot p_{1,2} &= 0 \,. \end{aligned}$$

Choose $p_{1,2} = 1$, then $p_{1,1} = \lambda_1$, and $\mathbf{p_1} = \left[\begin{smallmatrix} \lambda_1 \\ 1 \end{smallmatrix}\right]$. Similarly, $\mathbf{p_2} = \left[\begin{smallmatrix} \lambda_2 \\ 1 \end{smallmatrix}\right]$. Vectors $\mathbf{p_1}$ and $\mathbf{p_2}$ are guaranteed to be linearly independent as they were derived from two distinct eigenvalues, $\lambda_1$ and $\lambda_2$.

Thus, equation (2) translates to

$$\begin{aligned} \begin{bmatrix} a_k \\ a_{k-1} \end{bmatrix} &= \begin{bmatrix} A & B \\ 1 & 0 \end{bmatrix}^{k-1} \begin{bmatrix} a_1 \\ a_0 \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{p_1} & \mathbf{p_2} \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}^{k-1} \begin{bmatrix} \mathbf{p_1} & \mathbf{p_2} \end{bmatrix}^{-1} \begin{bmatrix} a_1 \\ a_0 \end{bmatrix} \\ &= \begin{bmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}^{k-1} \begin{bmatrix} \frac{1}{\lambda_1 - \lambda_2} & \frac{-\lambda_2}{\lambda_1 - \lambda_2} \\ \frac{-1}{\lambda_1 - \lambda_2} & \frac{\lambda_2}{\lambda_1 - \lambda_2} \end{bmatrix} \begin{bmatrix} a_1 \\ a_0 \end{bmatrix} \\ \begin{bmatrix} a_k \\ a_{k-1} \end{bmatrix} &= \begin{bmatrix} \sum_{j=0}^{k-1} \lambda_1^j \lambda_2^{k-1-j} & -\lambda_1 \lambda_2 \sum_{j=0}^{k-2} \lambda_1^j \lambda_2^{k-2-j} \\ \sum_{j=0}^{k-2} \lambda_1^j \lambda_2^{k-2-j} & -\lambda_1 \lambda_2 \sum_{j=0}^{k-3} \lambda_1^j \lambda_2^{k-3-j} \end{bmatrix} \begin{bmatrix} a_1 \\ a_0 \end{bmatrix} \,. \end{aligned} \tag{4}$$

Define the first two elements of the sequence to be $a_0 = 1, a_1 = \lambda_1$. The reason behind that definition will become clearer in a short while. If we let $k = 2$, we can find the third term of the sequence:[2]

$$\begin{aligned} \begin{bmatrix} a_2 \\ a_1 \end{bmatrix} &= \begin{bmatrix} \lambda_1 + \lambda_2 & -\lambda_1 \lambda_2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ 1 \end{bmatrix} \\ \begin{bmatrix} a_2 \\ a_1 \end{bmatrix} &= \begin{bmatrix} \lambda_1^2 \\ \lambda_1 \end{bmatrix} \,. \end{aligned}$$

Hence, the third element, $a_2$, is $\lambda_1^2$. Repeated iterations for different $k$s derive the sequence $1, \lambda_1, \lambda_1^2, \ldots, \lambda_1^n, \ldots$. The same, of course holds for the sequence $1, \lambda_2, \lambda_2^2, \ldots$. Whenever it is not needed to distinguish between the two sequences, we will drop the subscripts and refer to the two sequences collectively as

$$1, \lambda_*, \lambda_*^2, \ldots \,. \tag{5}$$

[2]Note that for $k = 2$ we get $\lambda_1 + \lambda_2 = A$ and $-\lambda_1 \lambda_2 = B$, effectively bringing us back to matrix $W$. Also, we define $\sum_0^{-1} = 0$.

To check that these sequences indeed satisfy relation (1), we need to pick three arbitrary consecutive terms and show that the relation holds. Let $a_{k-2} = \lambda_*^{k-2}$, $a_{k-1} = \lambda_*^{k-1}$, and $a_k = \lambda_*^k$. Relation (1) is then rewritten as follows:

$$\lambda_*^k = A \cdot \lambda_*^{k-1} + B \cdot \lambda_*^{k-2} \, .$$

Divide by $\lambda_*^{k-2}$ and regroup terms to receive

$$\lambda_*^2 - A\lambda_* - B = 0 \, .$$

But this is no different than the characteristic equation (3). We already know that $\lambda_1$ and $\lambda_2$ are the only possible solutions for that equation. So we have shown that the sequences (5) satisfy the recursive relation.

While these sequences are the only two to satisfy the characteristic equation, any linear combination of the two, in fact, satisfies the original recursive relation. To illustrate this fact, define a sequence $s_0, s_1, s_2, \ldots$, such that each $s_k = c_1\lambda_1^k + c_2\lambda_2^k$. Substituting in relation (1) yields

$$\begin{aligned} s_k &= A \cdot s_{k-1} + B \cdot s_{k-2} \\ c_1\lambda_1^k + c_2\lambda_2^k &= A \cdot (c_1\lambda_1^{k-1} + c_2\lambda_2^{k-1}) + B \cdot (c_1\lambda_1^{k-2} + c_2\lambda_2^{k-2}) \\ c_1\lambda_1^k + c_2\lambda_2^k &= c_1 \cdot (A\lambda_1^{k-1} + B\lambda_1^{k-2}) + c_2 \cdot (A\lambda_2^{k-1} + B\lambda_2^{k-2}) \\ c_1\lambda_1^k + c_2\lambda_2^k &= c_1\lambda_1^k + c_2\lambda_2^k \, . \end{aligned}$$

We see now that there are infinitely many ways to define a sequence that satisfies the recurrence relation. Our earlier choice of $a_0 = 1, a_1 = \lambda_*$ was somewhat arbitrary in that sense. However the sequences formed by that choice are the "simpleset" ones for a basis.

## 1.2 An Eigenvalue with a Multiplicity of Two

When the characteristic equation (3) has only one zero, we can no longer have two distinct eigenvalues and, therefore, no two linearly independent eigenvectors to form a basis. This scenario can happen only if the discriminant of the characteristic equation is zero, namely $A^2 + 4B = 0$. The following relations can be drawn from that fact:

$$\begin{aligned} B &= -\frac{A^2}{4} \\ \lambda &= \frac{A}{2} \, . \end{aligned} \tag{6}$$

We will return to these properties shortly.

You may recall that a similar problem of repeated roots occasionally arises in a second-order homogenous differential equation with constant coefficients. We will examine how this problem is solved for differential equations and attempt to find a similar solution for our case. As we shall see, the similarities are striking.

Consider a differential equation

$$f''(t) + A \cdot f'(t) + B \cdot f(t) = 0\,, \tag{7}$$

where $A$ and $B$ are real numbers, and $B \neq 0$. Leonard Euler showed that the solutions for equation (7) depend on its corresponding characteristic equation [[[CITE]]]

$$r^2 + Ar + B = 0\,.$$

If the characteristic equation has two real roots, $r_1$ and $r_2$, then the fundamental solutions for (7) are $e^{r_1 t}$ and $e^{r_2 t}$. These two solutions form a basis where every linear combination of the two is also a solution for (7). This result is analogous to the one we derived in the previous section.

Furthermore, it can be shown [[[CITE]]] that if the characteristic equation has only one root, $r$, then the fundamental solutions for (7) are $e^{rt}$ and $te^{rt}$. Since $t$ is an independent variable, rather than a constant, the two solutions are linearly independent.

This conclusion suggests that a similar approach might be applied in our case. Consider the sequence (5). If we assume that $e^r$ is analogous to $\lambda$ and $t$ is to the exponent, then a candidate for the second member in the basis is the sequence

$$\begin{aligned} &0\lambda^0, 1\lambda^1, 2\lambda^2, 3\lambda^3, \ldots \\ &= 0, \lambda, 2\lambda^2, 3\lambda^3, \ldots\,. \end{aligned} \tag{8}$$

We proceed by checking that the sequence satisfies relation (1) under the constraints set forth in (6). Pick three arbitrary consecutive terms, $a_{k-2} = (k-2)\lambda^{k-2}, a_{k-1} = (k-1)\lambda^{k-1}$, and $a_k = k\lambda^k$. For $k \geq 2$ we have

$$\begin{aligned} a_k &= A \cdot a_{k-1} + B \cdot a_{k-2} \\ k\lambda^k &= A(k-1)\lambda^{k-1} + B(k-2)\lambda^{k-2} \\ k\lambda^2 &= A(k-1)\lambda + B(k-2) \\ 0 &= -k(\lambda^2 - A\lambda - B) - A\lambda - 2B \\ 0 &= -\frac{A^2}{2} + \frac{A^2}{2}\,. \end{aligned}$$

Therefore, sequence (8) satisfies the recurrence relation. Additionally, any linear combination of the two sequences is also a solution.

## 1.3 Complex Conjugate Eigenvalues

The third case we need to consider is when the characteristic equation (3) has no real roots. In such a case, the two roots, $\lambda_1$ and $\lambda_2$ are a pair of complex conjugates. The real and imaginary parts of both eigenvalues are $\Re\lambda_* = \frac{A}{2}$ and $\Im\lambda_* = \pm\frac{\sqrt{-A^2-4B}}{2}$, respectively. The process of finding the eigenvectors of such a matrix is identical to the one presented in section 1.1, resulting in the eigenvectors $\mathbf{p_1}$ and $\mathbf{p_2}$. In fact, the relation between $a_k$ and $k$ is the one shown in equation (4) and brought here again for easy reference:

$$\begin{bmatrix} a_k \\ a_{k-1} \end{bmatrix} = \begin{bmatrix} \sum_{j=0}^{k-1} \lambda_1^j \lambda_2^{k-1-j} & -\lambda_1\lambda_2 \sum_{j=0}^{k-2} \lambda_1^j \lambda_2^{k-2-j} \\ \sum_{j=0}^{k-2} \lambda_1^j \lambda_2^{k-2-j} & -\lambda_1\lambda_2 \sum_{j=0}^{k-3} \lambda_1^j \lambda_2^{k-3-j} \end{bmatrix} \begin{bmatrix} a_1 \\ a_0 \end{bmatrix} . \tag{4}$$

Therefore, the sequences that satisfy relation (1) are

$$\begin{cases} 1, \Re\lambda_* + \Im\lambda_* i, (\Re\lambda_* + \Im\lambda_* i)^2, (\Re\lambda_* + \Im\lambda_* i)^3, \ldots, (\Re\lambda_* + \Im\lambda_* i)^k, \ldots \\ 1, \Re\lambda_* - \Im\lambda_* i, (\Re\lambda_* - \Im\lambda_* i)^2, (\Re\lambda_* - \Im\lambda_* i)^3, \ldots, (\Re\lambda_* - \Im\lambda_* i)^k, \ldots \end{cases} , \tag{9}$$

where $i^2 = -1$. Applying the *binomial theorem*, the $k$th term in each sequences can be rewritten as

$$\sum_{j=0}^{k} \binom{k}{j} (\Re\lambda_*)^j (\Im\lambda_*)^{k-j} i^{k-j} \quad \text{or}$$
$$\sum_{j=0}^{k} \binom{k}{j} (\Re\lambda_*)^j (-\Im\lambda_*)^{k-j} i^{k-j} .$$

Most terms in these two sequences contain both real and imaginary parts[3]. As we showed in the discussion in section 1.1, any linear combination of the two sequences also satisfies the recurrence relation. It might be of interest, therefore, to come up with a different basis in which one of the sequences is entirely real.

Such a basis is constructed if we add the two sequences in (9) to form a third sequence, $u_1, u_2, \ldots, u_k, \ldots$. We examine the $k$th term in that sequence:

$$\begin{aligned} u_k &= \sum_{j=0}^{k} \binom{k}{j} (\Re\lambda_*)^j (\Im\lambda_*)^{k-j} i^{k-j} + \binom{k}{j} (\Re\lambda_*)^j (-\Im\lambda_*)^{k-j} i^{k-j} \\ &= \sum_{j=0}^{k} \binom{k}{j} (\Re\lambda_*)^j i^{k-j} \left[ (\Im\lambda_*)^{k-j} + (-\Im\lambda_*)^{k-j} \right] . \end{aligned}$$

When the difference $k - j$ is odd, the two terms inside the brackets offset each other, and the product is zero. When the difference $k - j$ is even, then $i^{k-j}$ is guaranteed to be real. Therefore, each term $u_k$ is real.

Similarly, subtracting the two sequences in (9), derives a fourth sequence in which the terms are entirely imaginary.

[3]But be careful not to confuse the real and imaginary parts of $\lambda_*$ with the real and imaginary parts of the terms in the sequences; with the exception of the first two terms in each sequence, any term is a complex number that contains both $\Re\lambda_*$ and $\Im\lambda_*$ in its real and imaginary parts.

## 2 Two Case Studies

Next we turn to see some interesting properties of second-order linear homogenous recurrence relations with constant coefficients. We examine two cases: the Fibonacci sequence and the family of arithmetic sequences.

### 2.1 The Fibonacci Sequence

The Fibonacci sequence is a sequence in which each term is the sum of the two previous ones. Put it formally,

$$F_k = F_{k-1} + F_{k-2} \ , \tag{10}$$

for all integers $k \geq 2$. Additionally, the first two terms are defined, $F_0 = 0$ and $F_1 = 1$. Quite surprisingly, the sequence appears in nature and was also shown to have many applications in science. [[[CITE]]]

If we define each vector in the Fibonacci sequence as $\mathbf{F_k} = \left[\begin{smallmatrix} F_k \\ F_{k-1} \end{smallmatrix}\right]$, for all integers $k \geq 1$, we can represent this sequence as a system of two equations in a matrix form:

$$\begin{bmatrix} F_k \\ F_{k-1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} F_{k-1} \\ F_{k-2} \end{bmatrix} .$$

And equation (2) for this case becomes

$$\begin{bmatrix} F_k \\ F_{k-1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^{k-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix} .$$

Note that the transition matrix is symmetric. Therefore, applying the *spectral theorem*, we know that the eigenvectors for it are orthogonal. We continue by finding the eigenvalues and eigenvectors. The characteristic equation for the Fibonacci sequence is

$$\lambda^2 - \lambda - 1 = 0 .$$

The are two solutions for the characteristic equation: $\lambda_1 = \frac{1+\sqrt{5}}{2}$ and $\lambda_2 = \frac{1-\sqrt{5}}{2}$. The corresponding eigenvectors are $\mathbf{p_1} = \left[\begin{smallmatrix} \lambda_1 \\ 1 \end{smallmatrix}\right]$ and $\mathbf{p_2} \left[\begin{smallmatrix} \lambda_2 \\ 1 \end{smallmatrix}\right]$.

In figure 1, we present the eigenspace for the Fibonacci sequence. Note that the two eigenvectors are orthogonal as the spectral theorem assured. Next, we need to consider to issue of *stability*.

Recall that if a matrix $A$ has an eigenvalue $\lambda$ and a corresponding eignevector $\mathbf{v}$, then $A\mathbf{v} = \lambda\mathbf{v}$. The underlying concept is that when the linear transformation $A$ is applied to an eigenvector $\mathbf{v}$ it scales it by its corresponding eigenvalue $\lambda$. Thus, each eigenvector spans a one-dimensional subspace[4] which, in the context of this discussion, we call an *equilibrium state*. This equilibrium state is completely analogous to the equilibrium state in differential equations.

[4]Note that, by definition, the zero vector is *not* an eigenvector. Hence, the term *subspace* may be slightly misleading, if not inaccurate.

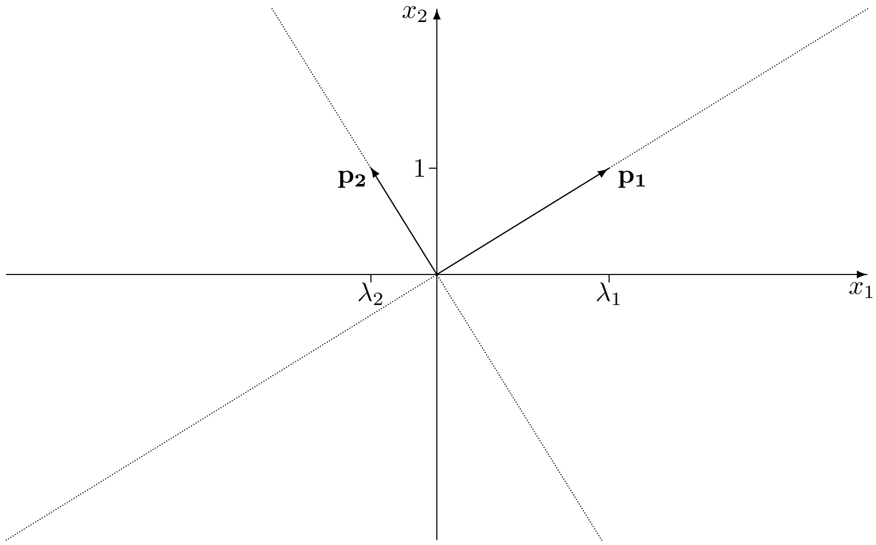


Figure 1: Eigenspace for the Fibonacci Sequence

Generally speaking, equilibrium states can be stable, partially stable, or unstable. In all cases, once a vector is in the equilibrium state, repeated transformation will ensure that it stays there. The three equilibrium states differ, however, in how vectors in their vicinity are transformed. A vector near a stable equilibrium will be "pushed"[5] toward the equilibrium state, though it will never reach it. A vector near an unstable equilibrium will be "pushed" away from the equilibrium state. Lastly, a vector near a partially stable state would be "pushed" either away or toward the equilibrium state, depending on its location relative to the equilibrium state.

In our case, $\lambda_1$ is positive, meaning that any vector on or around the equilibrium state spanned by $\mathbf{p_1}$ will be dilated, or scaled up, away from the origin. Vectors that are closer to the equilibrium state will be more affected than those farther away from it. The direction in which the vectors are dilated is determined by their dot product with $\mathbf{p_1}$: those with a positive dot product tend toward $\mathbf{p_1}$, and those with a negative dot product tend toward $-\mathbf{p_1}$. In particular, vectors to the right of the subspace spanned by $\mathbf{p_2}$ will steer further away of it to the right, and those to the left of it will steer further away to the left. As mentioned earlier, vectors on the subspace spanned by $\mathbf{p_2}$ will stay there. Put it formally, $\mathbf{p_2}$ spans an unstable equilibrium state.

In contrast, $\lambda_2$ is negative. In this case, vectors that are not orthogonal to $\mathbf{p_2}$ are shrunk, or scaled down, toward the origin. That is to say, any vector that is not on the subspace spanned by $\mathbf{p_1}$ will approach it. Formally, $\mathbf{p_1}$ spans

[5]By *"pushed"* we mean the change in the distance between the vector and the equilibrium state, once the linear transformation is applied on the vector. This distance is measured by the difference between the vector and its projection onto the equilibrium state subspace.

a stable equilibrium state.

Figure 2 illustrates the previous discussion pictorially. We show the first four terms of the Fibonacci sequence as vecotrs. Note how each subsequent vector approaches the subspace spanned by $\mathbf{p_1}$ while oscillating about it. This characteristic is due to fact that $\lambda_2$ is negative. Also, the magnitude of each subsequent vector is increasing. This is due to the fact that $\lambda_1$ is positive.

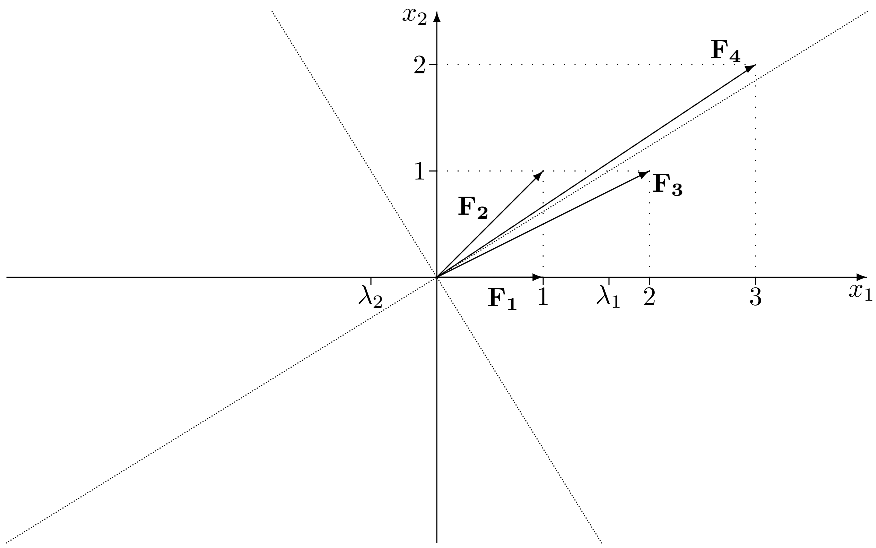


Figure 2: Four Terms of the Fibonacci Sequence

## 2.2 Arithmetic Sequences

In our second example we discuss the following sequence:

$$G_k = 2G_{k-1} - G_{k-2} \,, \tag{11}$$

for all integers $k \geq 2$. We start by defining the first two terms as $G_0 = 1$ and $G_1 = 1$. First observe that if $G_1 = G_0$ then $G_2 = G_0$, and indeed $G_k = G_0$ for all integers $k > 0$.

Similar to the previous discussion, define $\mathbf{G_k} = \begin{bmatrix} G_k \\ G_{k-1} \end{bmatrix}$, for all integers $k \geq 1$. We can represent the sequence in (11) as a system of two equations in a matrix form:

$$\begin{bmatrix} G_k \\ G_{k-1} \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} G_{k-1} \\ G_{k-2} \end{bmatrix} .$$

Equation (2) for this case becomes

$$\begin{bmatrix} G_k \\ G_{k-1} \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}^{k-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix} .$$

If we proceed to find the eigenvalues of this system, we find that the characteristic equation, $\lambda^2 - 2\lambda + 1 = 0$, has only one root $\lambda = 1$. This is an example of an eigenvalue with a multiplicity of two. Finding two linearly independent eigenvectors is therefore impossible.

Eigenvector $\mathbf{p} = \left[\begin{smallmatrix} p_1 \\ p_2 \end{smallmatrix}\right]$ has to satisfy the equation

$$\begin{aligned} 1 \cdot p_1 - \lambda \cdot p_2 &= 0 \\ p_1 - p_2 &= 0\,. \end{aligned}$$

Choose $p_1$=1, then $p_2$=1, and $\mathbf{p} = \left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right]$.

Remember from equation (8) that a second, linearly independent sequence that satisfies relation (11) is

$$0, \lambda, 2\lambda^2, 3\lambda^3, \ldots\,,$$

which in the case of $\lambda = 1$ becomes

$$0, 1, 2, 3, \ldots\,.$$

Furthermore, we showed that any linear combination of the sequences

$$\begin{cases} 1, 1, 1, 1, \ldots \\ 0, 1, 2, 3, \ldots \end{cases} \tag{12}$$

also satisfies relation (11).

Note that any arithmetic sequence can be written as linear combinations of the above sequences. If $a_k = a_{k-1} + c$ for integers $k \geq 1$ and $a_0 = d$, then $a_k = c \cdot k + d \cdot 1$, or the second sequence in (12) multiplied by $c$ added to the first sequence multiplied by $d$. Therefore, we have shown that any arithmetic sequence satisfies the recurrence relation given in (11).