Classification of the Irreducible Representations of the Dihedral Group D_{2n}

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Let D_{2n} be the dihedral group with $2n$ elements, where $n \geq 3$, corresponding to rigid transformation of a regular n-gon. In this paper, we classify the irreducible representations of D_{2n} and their corresponding irreducible D_{2n}-modules.

Typically, one writes the presentation $D_{2n} = \langle a, b \mid a^n = b^2 = 1, b^{-1}ab = a^{-1} \rangle$, where the generator a corresponds to a clockwise rotation of $\frac{2\pi}{n}$ and the generator b corresponds to a reflection over a fixed axis through one of the vertices. It turns out that for a our purpose it is more convenient to think of D_{2n} with the presentation $\langle s_1, s_2 \mid s_1^2 = s_2^2 = (s_1 s_2)^n = 1 \rangle$.

Theorem. *Let D_{2n} be the dihedral group with $2n$ elements. We distinguish two cases:*

1. *$n = 2k + 1$ is odd. Up to equivalency of representations, the irreducible representations of D_{2n} are*

$$\begin{array}{ccc} \rho_0 : D_{2n} \to GL_1(\mathbb{C}) & \rho_{-1} : D_{2n} \to GL_1(\mathbb{C}) & \rho_i : D_{2n} \to GL_2(\mathbb{C}) \\ s_1 \mapsto (1) & s_1 \mapsto (-1) & s_1 \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ s_2 \mapsto (1) & s_2 \mapsto (-1) & s_2 \mapsto \begin{pmatrix} 0 & \omega_n^{-i} \\ \omega_n^i & 0 \end{pmatrix} \end{array},$$

   where $\omega_n = e^{\frac{2\pi}{n}}$ is the principal nth root of unity, and $i : 1, \ldots, k$.

   The corresponding D_{2n}-modules are $V_{-1}, \ldots V_k$, such that if $\{v^{(0)}\}$, $\{v^{(-1)}\}$, and $\{v_1^{(i)}, v_2^{(i)}\}$ are the bases for V_0, V_{-1}, and V_i, respectively, and where $i : 1, \ldots, k$, then the following actions[1] *define the D_{2n}-modules:*

$$\begin{cases} v^{(0)} \circ s_1 = v^{(0)} \\ v^{(0)} \circ s_2 = v^{(0)} \end{cases} \quad \begin{cases} v^{(-1)} \circ s_1 = -v^{(-1)} \\ v^{(-1)} \circ s_2 = -v^{(-1)} \end{cases} \quad \begin{cases} v_1^{(i)} \circ s_1 = v_2^{(i)} & v_2^{(i)} \circ s_1 = v_1^{(i)} \\ v_1^{(i)} \circ s_2 = \omega_n^{-i} v_2^{(i)} & v_2^{(i)} \circ s_2 = \omega_n^{i} v_1^{(i)} \end{cases}.$$

[1] With a slight abuse of notation, we denote the actions $V_i \times D_{2n} \to V_i$ by $\circ$ for all $i : -1, \ldots, k$.

2. $n = 2k$ is even. Up to equivalency of representations, the irreducible representations of D_{2n} are

$$\begin{array}{cccc} \rho_0 : D_{2n} \to GL_1(\mathbb{C}) & \rho_{-1} : D_{2n} \to GL_1(\mathbb{C}) & \rho_{-2} : D_{2n} \to GL_1(\mathbb{C}) & \rho_{-3} : D_{2n} \to GL_1(\mathbb{C}) \\ s_1 \mapsto (1) & s_1 \mapsto (1) & s_1 \mapsto (-1) & s_1 \mapsto (-1) \\ s_2 \mapsto (1) & s_2 \mapsto (-1) & s_2 \mapsto (1) & s_2 \mapsto (-1) \end{array}$$

$$\begin{aligned} \rho_i : D_{2n} &\to GL_2(\mathbb{C}) \\ s_1 &\mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ s_2 &\mapsto \begin{pmatrix} 0 & \omega_n^{-i} \\ \omega_n^i & 0 \end{pmatrix}, \end{aligned}$$

where $\omega_n = e^{\frac{2\pi}{n}}$ is the principal nth root of unity, and $i : 1, \dots, k-1$.

The corresponding D_{2n}-modules are $V_{-3}, \dots V_{k-1}$, such that if $\{v^{(0)}\}$, $\{v^{(-1)}\}$, $\{v^{(-2)}\}$, $\{v^{(-3)}\}$, and $\{v_1^{(i)}, v_2^{(i)}\}$ are the bases for V_0, V_{-1}, V_{-2}, V_{-3}, and V_i, respectively, and where i : $1, \dots, k-1$, then the following actions define the D_{2n}-modules:

$$\begin{cases} v^{(0)} \circ s_1 = v^{(0)} \\ v^{(0)} \circ s_2 = v^{(0)} \end{cases} \quad \begin{cases} v^{(-1)} \circ s_1 = v^{(-1)} \\ v^{(-1)} \circ s_2 = -v^{(-1)} \end{cases} \quad \begin{cases} v^{(-2)} \circ s_1 = -v^{(-2)} \\ v^{(-2)} \circ s_2 = v^{(-2)} \end{cases} \quad \begin{cases} v^{(-3)} \circ s_1 = -v^{(-3)} \\ v^{(-3)} \circ s_2 = -v^{(-3)} \end{cases}$$

$$\begin{cases} v_1^{(i)} \circ s_1 = v_2^{(i)} & v_2^{(i)} \circ s_1 = v_1^{(i)} \\ v_1^{(i)} \circ s_2 = \omega_n^{-i} v_2^{(i)} & v_2^{(i)} \circ s_2 = \omega_n^{i} v_1^{(i)} \end{cases}.$$

Before we proceed with the proof of the theorem, we outline the crucial steps of the proof. We will show that (i) the above maps are indeed representations of D_{2n}; (ii) they are irreducible; (iii) no two of them are equivalent; and (iv) they exhaust the list of non-equivalent D_{2n} representations. The proofs of (ii) and (iii) will not be directly on the representations ρ_i, but rather on the corresponding D_{2n}-modules. Such a method is valid since a representation of D_{2n} is irreducible if and only if its corresponding D_{2n}-module is irreducible, and two representations are equivalent if and only if their corresponding D_{2n}-modules are isomorphic. The proof of (iv) will also look at the corresponding D_{2n}-modules and will invoke a counting argument.

Since the statement of the theorem splits into two cases—when n is odd and when n is even—we will also consider the two cases separately. Fortunately, most of the arguments for one case are applicable to the other one with only a few minor changes.

Proof. Suppose $n = 2k+1$ is odd. Our first step (i) of the proof is showing that the maps ρ_i are representations. To that end, we need to check that the ρ_is are group homomorphisms. It suffices to check that the relations on the generators s_1 and s_2 are preserved under the maps. That is, we need to verify that $(s_1)\rho_i(s_1)\rho_i = (s_2)\rho_i(s_2)\rho_i = I$ and $[(s_1)\rho_i(s_2)\rho_i]^k (s_1)\rho_i = (s_2)\rho_i [(s_1)\rho_i(s_2)\rho_i]^k$, for $i = -1, \dots, k$ and where I is the identity matrix.

Consider first $i = 0$. Then $(s_1)\rho_0(s_1)\rho_0 = (1)(1) = (1)$ and $(s_2)\rho_0(s_2)\rho_0 = (1)(1) = (1)$.

In addition, $[(s_1)\rho_0(s_2)\rho_0]^k (s_1)\rho_0 = (1) = (s_2)\rho_0 [(s_1)\rho_0(s_2)\rho_0]^k$. Similarly, if $i = -1$, then $(s_1)\rho_{-1}(s_1)\rho_{-1} = (-1)(-1) = (1)$ and $(s_2)\rho_{-1}(s_2)\rho_{-1} = (-1)(-1) = (1)$. As before, we also have $[(s_1)\rho_{-1}(s_2)\rho_{-1}]^k (s_1)\rho_{-1} = (-1) = (s_2)\rho_{-1} [(s_1)\rho_{-1}(s_2)\rho_{-1}]^k$. Consider now $1 \le i \le k$. Clearly, we have

$$(s_1)\rho_i(s_1)\rho_i = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad (s_2)\rho_i(s_2)\rho_i = \begin{pmatrix} 0 & \omega_n^{-i} \\ \omega_n^i & 0 \end{pmatrix} \begin{pmatrix} 0 & \omega_n^{-i} \\ \omega_n^i & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} .$$

For the alternating product, note first that $i(2k+1) \equiv 0 \pmod n$, so $ik \equiv -i-ik \pmod n$. Hence, $\omega_n^{ik} = \omega_n^{-i-ik}$, so we have

$$\begin{aligned}
[(s_1)\rho_i(s_2)\rho_i]^k (s_1)\rho_i &= \left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & \omega_n^{-i} \\ \omega_n^i & 0 \end{pmatrix}\right]^k \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \left[\begin{pmatrix} \omega_n^i & 0 \\ 0 & \omega_n^{-i} \end{pmatrix}\right]^k \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \omega_n^{ik} \\ \omega_n^{-ik} & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & \omega_n^{-i-ik} \\ \omega_n^{i+ik} & 0 \end{pmatrix} = \begin{pmatrix} 0 & \omega_n^{-i} \\ \omega_n^i & 0 \end{pmatrix} \left[\begin{pmatrix} \omega_n^i & 0 \\ 0 & \omega_n^{-i} \end{pmatrix}\right]^k \\
&= \begin{pmatrix} 0 & \omega_n^{-i} \\ \omega_n^i & 0 \end{pmatrix} \left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & \omega_n^{-i} \\ \omega_n^i & 0 \end{pmatrix}\right]^k = (s_2)\rho_i [(s_1)\rho_i(s_2)\rho_i]^k .
\end{aligned}$$

We conclude that the maps above are indeed representations.

Next (ii) we show each map ρ_i is irreducible, where $i : -1, \ldots, k$. Recall that by definition a representation ρ of D_{2n} is irreducible if and only if its corresponding D_{2n}-module with respect to some fixed basis is irreducible. The dimension of a nonzero submodule $W \subsetneq V$ must be strictly less than the dimension of V and strictly greater than zero. The D_{2n}-modules V_{-1} and V_0 are one-dimensional, so clearly they are irreducible.

It remains to check the V_i is irreducible for $i : 1, \ldots, k$. Note that if a two-dimensional D_{2n}-module V_i is reducible, then it must have a one-dimensional D_{2n}-submodule W. That is, $W = \mathbb{C}\text{-span}\{\alpha v_1^{(i)} + \beta v_2^{(i)}\}$, for some complex numbers α and β, where at least one of α, β is nonzero. It follows that there exists a nonzero complex number γ for which

$$\left(\alpha v_1^{(i)} + \beta v_2^{(i)}\right) \circ s_1 = \beta v_1^{(i)} + \alpha v_2^{(i)} = \gamma \left(\alpha v_1^{(i)} + \beta v_2^{(i)}\right) .$$

Hence, $\alpha = \beta$ and we may choose $\alpha = \beta = 1$ for simplicity. Since W is a D_{2n}-submodule, there exists a nonzero complex number δ for which

$$\left(v_1^{(i)} + v_2^{(i)}\right) \circ s_2 = \omega_n^i v_1^{(i)} + \omega_n^{-i} v_2^{(i)} = \delta \left(v_1^{(i)} + v_2^{(i)}\right) .$$

This is possible if and only if $\omega_n^i = \omega_n^{-i}$. But this is the case if and only if $i \equiv -i \pmod n$, which is equivalent to requiring $2i \equiv 0 \pmod n$. But since $2 \le 2i \le n-1$, we conclude that a one-dimensional D_{2n}-module W does not exist. We have shown that the representations ρ_i are irreducible for $i : -1, \ldots, k$.

Next (iii) we show that ρ_i and ρ_j are not equivalent for $i \neq j$ and where $-1 \leq i, j \leq k$. A necessary condition for two representations $\rho : G \to GL_n(\mathbb{C})$ and $\sigma : G \to GL_m(\mathbb{C})$ is $n = m$, so we need only show that ρ_{-1} is not equivalent to ρ_0 and that ρ_i is not equivalent to ρ_j for $i \neq j$ and $1 \leq i, j \leq k$.

By definition, ρ_{-1} is equivalent to ρ_0 if there exists a 1×1 invertible matrix T such that for all $s \in D_{2n}$, we have $(s)\rho_{-1} = T^{-1}(s)\rho_0 T$. In particular, for these two representations to be equivalent, there must be a nonzero complex number α such that

$$(-1) = (s_1)\rho_{-1} = (\alpha)^{-1}(s)\rho_0(\alpha) = (\alpha)^{-1}(1)(\alpha) = (\alpha)^{-1}(\alpha)(1) = (1)\,,$$

which clearly is impossible.

Consider now $i \neq j$ where $1 \leq i, j \leq k$. Recall that ρ_i and ρ_j are equivalent if and only if their corresponding D_{2n}-modules with respect to some fixed bases are isomorphic. Consider a D_{2n}-module homomorphism $\varphi : V_i \to V_j$ with $v_1^{(i)} \mapsto \alpha v_1^{(j)} + \beta v_2^{(j)}$ and $v_2^{(i)} \mapsto \gamma v_1^{(j)} + \delta v_2^{(j)}$ for some complex numbers α, β, γ, and δ. Then

$$v_2^{(i)} = v_1^{(i)} \circ s_1 \mapsto \alpha v_1^{(j)} \circ s_1 + \beta v_2^{(j)} \circ s_1 = \beta v_1^{(j)} + \alpha v_2^{(j)} \qquad \text{and} \qquad v_2^{(i)} \mapsto \gamma v_1^{(j)} + \delta v_2^{(j)}\,.$$

Hence, $\alpha = \delta$ and $\beta = \gamma$, so that we may write $v_2^{(i)} \mapsto \beta v_1^{(j)} + \alpha v_2^{(j)}$. Also,

$$\omega_n^{-i} v_2^{(i)} = v_1^{(i)} \circ s_2 \mapsto \alpha v_1^{(j)} \circ s_2 + \beta v_2^{(j)} \circ s_2 = \beta \omega_n^{j} v_1^{(j)} + \alpha \omega_n^{-j} v_2^{(j)} \quad \text{and} \quad \omega_n^{-i} v_2^{(i)} \mapsto \beta \omega_n^{-i} v_1^{(j)} + \alpha \omega_n^{-i} v_2^{(j)}\,.$$

Thus, we must have $\beta\omega_n^{j} = \beta\omega_n^{-i}$ and $\alpha\omega_n^{-j} = \alpha\omega_n^{-i}$. This is possible if and only if ($\alpha = 0$ or $i \equiv j \pmod{n}$) and ($\beta = 0$ or $i \equiv -j \pmod{n}$). Since $1 \leq i, j \leq k = \frac{n-1}{2}$ and $i \neq j$, we must have $\alpha = \beta = 0$. It follows that φ is the zero D_{2n}-module homomorphism, which is not invertible. In particular, φ is not a D_{2n}-module isomorphism. We conclude that ρ_i is not equivalent to ρ_j for all $i \neq j$ where $1 \leq i, j \leq k$.

The last (iv) remaining step is showing that the ρ_is we have defined above exhaust the list of non-equivalent D_{2n} representations. Here we recall that if $V_{-1}, \ldots V_k$ form a complete set of non-isomorphic irreducible D_{2n}-submodules, then

$$\sum_{i=-1}^{k} (\dim V_i)^2 = |D_{2n}|\,.$$

But, $\sum_{i=-1}^{k} (\dim V_i)^2 = 2(1^2) + k(2^2) = 2(2k+1) = 2n = |D_{2n}|$, to complete the classification in the case of n odd.

We now turn to the case $n = 2k$ even. We start with step (i): showing that $\rho_{-3}, \ldots, \rho_{k-1}$ are representations. We again need to check the relations on the images of the generators: $(s_1)\rho_i(s_1)\rho_i = (s_2)\rho_i(s_2)\rho_i = I$ and $[(s_1)\rho_i(s_2)\rho_i]^k = [(s_2)\rho_i(s_1)\rho_i]^k\,(s_2)\rho_i$, for $i = -3, \ldots, k-1$ and where I is the identity matrix.

Consider first $-3 \leq i \leq 0$. Then $(s_1)\rho_i(s_1)\rho_i = (\pm 1)(\pm 1) = (1)$ and $(s_2)\rho_i(s_2)\rho_i = (\pm 1)(\pm 1) =$

(1). In addition, $[(s_1)\rho_i(s_2)\rho_i]^k = (\pm 1)^k = [(s_2)\rho_i(s_1)\rho_i]^k$. Consider now $1 \leq i \leq k-1$. Then $(s_1)\rho_i(s_1)\rho_i = (s_2)\rho_i(s_2)\rho_i = I$ is exactly as in the case of n odd. For the alternating product, we have $2ik \equiv 0 \pmod{n}$, so $ik \equiv -ik \pmod{n}$. Hence, $\omega_n^{ik} = \omega_n^{-ik}$, and we have

$$[(s_1)\rho_i(s_2)\rho_i]^k = \left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & \omega_n^{-i} \\ \omega_n^i & 0 \end{pmatrix}\right]^k = \left[\begin{pmatrix} \omega_n^i & 0 \\ 0 & \omega_n^{-i} \end{pmatrix}\right]^k = \begin{pmatrix} \omega_n^{ik} & 0 \\ 0 & \omega_n^{-ik} \end{pmatrix} = \begin{pmatrix} \omega_n^{-ik} & 0 \\ 0 & \omega_n^{ik} \end{pmatrix}$$
$$= \left[\begin{pmatrix} \omega_n^{-i} & 0 \\ 0 & \omega_n^{i} \end{pmatrix}\right]^k = \left[\begin{pmatrix} 0 & \omega_n^{-i} \\ \omega_n^i & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right]^k = [(s_2)\rho_i(s_1)\rho_i]^k \ .$$

As before, we conclude that the maps above are indeed representations.

Next (ii), we need to show that the representations ρ_i are irreducible, where $i : -3, \dots, k-1$. The corresponding D_{2n}-modules $V_{-3}, \dots, V_0$ are one-dimensionals and thus irreducible. Following the exact same argument as in the case n odd, the remaining ρ_i are also irreducible provided that $2i \not\equiv 0 \pmod{n}$. But since $2 \leq 2i \leq 2(k-1) = n-2$, this is indeed the case.

Next (iii), we show that ρ_i and ρ_j are not equivalent for $i \neq j$ and $-3 \leq i, j \leq k-1$. For $\rho_{-3}, \dots, \rho_0$, using again the fact that 1×1 matrices commute, we know that any two representations ρ_i and ρ_j are equivalent if $(s_1)\rho_i = \rho_j$ and $(s_2)\rho_i = \rho_j$. But this does not hold for any such pair ρ_i, ρ_j. Consider now $1 \leq i, j \leq k-1$. Following the same argument as in the case n odd, two representations ρ_i and ρ_j are equivalent provided that $i \equiv j \pmod{n}$ or $i \equiv -j \pmod{n}$. But again, since $1 \leq i, j \leq k-1 = \frac{n-2}{2}$ and $i \neq j$, we conclude that ρ_i is not equivalent to ρ_j for all $i \neq j$ and $1 \leq i, j \leq k-1$.

Lastly (iv), we note that $\sum_{i=-3}^{k-1} = 4(1^2) + (k-1)(2^2) = 4k = 2n = |D_{2n}|$, so our list of non-equivalent D_{2n} representations is complete. This concludes the proof for the case n even and proves the theorem. $\square$