Classification of the Irreducible Modules of the Monomial Matrix Group $G(2,1,n)$

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Let $G(2,1,n) \cong C_2 \wr S_n$, where $n \geq 2$, be the monomial matrix group of $n \times n$ matrices with exactly one nonzero entry in every column and row, and the nonzero entries are in C_2, the cyclic group of order two. In this paper, we classify the irreducible representations of $G(2,1,n)$.

Recall that $G(2,1,n)$ is generated by $\{e_1, s_1, \ldots, s_{n-1}\}$ with the following relations:

$$\begin{aligned} &e_1 s_1 e_1 s_1 = s_1 e_1 s_1 e_1, \qquad s_i s_j = s_j s_i \text{ for } |i-j| \geq 1, \qquad s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}, \\ &e_1^2 = s_1^2 = \cdots = s_{n-1}^2 = 1\,. \end{aligned} \tag{1}$$

It is often more convenient, however, to define the elements $e_j = s_{j-1} e_{j-1} s_{j-1}$ for $j = 2, \ldots, n$ and to think of $G(2,1,n)$ as generated by $\{e_1, \ldots, e_n, s_1, \ldots, s_{n-1}\}$ with the relations

$$\begin{aligned} &e_i e_j = e_j e_i, \qquad e_i s_j = s_j e_{i s_j}, \qquad s_i s_j = s_j s_i \text{ for } |i-j| \geq 1, \qquad s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}, \\ &e_1^2 = s_1^2 = \cdots = s_{n-1}^2 = 1\,. \end{aligned}$$

We shall switch back and forth between the two presentations of $G(2,1,n)$ in the following discussion.

Before we can state the classification theorem, we must extend the definitions of a *tableau of shape* λ and a *standard tableau of shape* λ to an ordered pair $\lambda = (\lambda^{(1)}, \lambda^{(2)})$ of partitions with n total boxes. We also provide a natural extension to the definition of the *tableau content* $C(T(i))$ of the box containing i in a standard tableau T of shape λ to the case when λ is an ordered pair $(\lambda^{(1)}, \lambda^{(2)})$ of partitions with n total boxes.

Definition 1. A *tableau of shape* λ, where λ is an order pair $(\lambda^{(1)}, \lambda^{(2)})$ of partitions with n total boxes, is a filling of the boxes of λ by $1, 2, \ldots, |\lambda^{(1)}| + |\lambda^{(2)}|$.

Definition 2. Let T be a tableau of shape $\lambda = (\lambda^{(1)}, \lambda^{(2)})$. Then T is said to be a *standard tableau of shape* λ if the entries in the fillings of $\lambda^{(1)}$ and $\lambda^{(2)}$ each increase in rows and columns.

Definition 3. Let T be a standard tableau of shape $\lambda = (\lambda^{(1)}, \lambda^{(2)})$. The *tableau content* $C(T(i))$ of the box containing i in T is the content $c_{\lambda^{(1)}}\left(\boxed{i}\right)$ if the box containing i is in the filling of $\lambda^{(1)}$ and $c_{\lambda^{(2)}}\left(\boxed{i}\right)$ otherwise.

We can now state the main theorem of this paper.

Theorem 4. *Let $G(2,1,n)$ be the monomial matrix group of $n \times n$ matrices with exactly one nonzero entry in every column and row, and the nonzero entries are in C_2, the cyclic group of order two.*

1. *The irreducible $G(2,1,n)$-modules V^λ are indexed by ordered pairs $\lambda = (\lambda^{(1)}, \lambda^{(2)})$ of partitions with n total boxes.*

2. *Their dimensions are $\dim(V^\lambda) = n_\lambda$, where n_λ is the number of standard tableaux of shape λ.*

3. *If $V^\lambda = \mathbb{C}\text{-span}\{v_T \mid T$ is a standard tableau of shape $\lambda\}$, then*

$$v_T e_i = (-1)^{m+1} v_T \ ,$$
$$v_T s_i = C_T(i) v_T + (1 + C_T(i)) v_{Ts_i} \ ,$$

where (i) the box containing i is in the partition $\lambda^{(m)}$; (ii) $v_{Ts_i} = 0$ if v_{Ts_i} is not a standard tableau; and (iii) $C_T(i) = \frac{1}{C(T(i+1)) - C(T(i))}$ if the boxes containing i and $i+1$ are in the same partition $\lambda^{(k)}$ and zero otherwise.

Before we proceed with the proof of the theorem, we outline the crucial steps of the proof. We will show that (i) the V^λ are indeed $G(2,1,n)$-modules; (ii) any two different such modules are non-isomorphic; (iii) these modules are irreducible; and (iv) this classification exhausts the list of irreducible $G(2,1,n)$-modules, up to isomorphism.

To prove (i), we will check that the relations of $G(2,1,n)$ are preserved under the action of the group on V^λ. The proof of (ii) will rely on generating elements analogous to the Murphy elements. We will show that their actions on a standard tableau completely determine the filling of the tableau. An immediate result of this would be (ii). For (iii), we will show that V^λ does not contain nonzero submodules besides itself by deriving some properties of the standard tableaux. Lastly, the proof of (iv) will involve a counting argument.

The following general observation will assist us through out the rest of the proof: The definition of a standard tableau implies that if neither of the boxes immediately to the right and immediately below a box containing i contain $i+1$, then they contain values greater than $i+1$. Moreover, the box containing $i+1$ is either in a different partitions $\lambda^{(k)}$ or it is strictly below and strictly to the right of the box containing i. In either case, the boxes above and to the left of the box containing $i+1$ contain values less than i. In such a case, the tableau resulting from permuting the boxes containing i and $i+1$ is a standard tableau.

We summarize this result with the following remark:

Remark 5. Let $\lambda = (\lambda^{(1)}, \lambda^{(2)})$ be an order pair of partitions with n total boxes, and let T be a standard partition of shape λ. Then whenever the boxes containing i and $i+1$ in T are not adjacent, Ts_i is also a standard tableau of shape λ. In particular, if the boxes containing i and $i+1$ are in different partitions $\lambda^{(1)}$ and $\lambda^{(2)}$, then Ts_i is a standard tableau of shape λ.

Proving Theorem 4 will be significantly easier to follow if we establish a few lemmas to help us later. The first lemma proves item (i) of the outline we presented above: the V^λ are $G(2,1,n)$-modules. The proof is rather long and tedious, though not particularly involved. In the interest of a smooth reading flow, the reader may wish the assume the lemma for the time being and return to its proof later.

Lemma 6. *Let $\lambda = (\lambda^{(1)}, \lambda^{(2)})$ be an order pair of partitions with n total boxes. Then $V^\lambda = \mathbb{C}\text{-span}\{v_T \mid T$ is a standard tableau of shape $\lambda\}$ is a $G(2,1,n)$-module with respect to the actions defined in Theorem 4(3).*

Proof. We need to check that the relations defined in (1) are satisfied by the speficied actions. In the rest of the proof, let $v_T \in V^\lambda$ for a standard tableau T.

1. The relation $e_1^2 = 1$

We start by showing that $v_T e_1^2 = v_T$. Suppose first that the box containing 1 is in $\lambda^{(1)}$. Then $v_T e_1^2 = v_T e_1 = v_T$. Otherwise, the box containing 1 is in $\lambda^{(2)}$. Then it follows that $v_T e_1^2 = -v_T e_i = v_T$, as needed.

2. The relation $s_i^2 = 1$

Next, we show that $v_T s_i^2 = v_T$ for $1 \le i \le n-1$. We consider the following cases regarding the relative positions of the boxes containing i and $i+1$ in T:

(a) the two boxes are adjacent;

(b) the two boxes are not adjacent.

In case (a), we have $C_T(i) = \pm 1$ and Ts_i is not standard. Thus, $v_T s_i^2 = \pm v_T s_i = v_T$. In case (b), we recall from Remark 5 that Ts_i is standard. We also clearly have $C_{Ts_i}(i) = -C_T(i)$. Hence,

$$\begin{aligned} v_T s_i^2 &= C_T(i) v_T s_i + \big[1 + C_T(i)\big] v_{Ts_i} s_i \\ &= C_T(i)^2 v_T + C_T(i)\big[1 + C_T(i)\big] v_{Ts_i} - C_T(i)\big[1 + C_T(i)\big] v_{Ts_i} + \big[1 - C_T(i)\big]\big[1 + C_T(i)\big] v_T = v_T \,. \end{aligned}$$

3. The relation $s_is_j = s_js_i$

Next, we show that $v_Ts_is_j = v_Ts_js_i$ for $1 \leq i,j \leq n-1$ and $|i-j| > 1$. We observe that $C_T(i) = C_{Ts_j}(i)$ and $C_T(j) = C_{Ts_i}(j)$. Thus,

$$\begin{aligned} v_Ts_is_j &= C_T(i)v_Ts_j + \big[1 + C_T(i)\big]v_{Ts_i}s_j \\ &= C_T(i)C_T(j)v_T + C_T(i)\big[1 + C_T(j)\big]v_{Ts_j} + \big[1 + C_T(i)\big]C_T(j)v_{Ts_i} + \big[1 + C_T(i)\big]\big[1 + C_T(j)\big]v_{Ts_is_j} \\ &= C_T(j)C_T(i)v_T + C_T(j)\big[1 + C_T(i)\big]v_{Ts_i} + \big[1 + C_T(j)\big]C_T(i)v_{Ts_j} + \big[1 + C_T(j)\big]\big[1 + C_T(i)\big]v_{Ts_js_i} \\ &= C_T(j)v_Ts_i + \big[1 + C_T(j)\big]v_{Ts_j}s_i = v_Ts_js_i \ , \end{aligned}$$

where we recall that for any $\sigma \in S_n$, we define $v_{T\sigma} = 0$ if $v_{T\sigma}$ is not a standard tableau.

4. The relation $s_1e_1s_1e_1 = e_1s_1e_1s_1$

The next relation we establish is $v_Ts_1e_1s_1e_1 = v_Te_1s_1e_1s_1$. The box containing 1 must be at the top left corner of one of the partitions $\lambda^{(k)}$. The box containing 2 can only be underneath the box containing 1, to the right of the box containing 1, or at the top left corner of the other partition $\lambda^{(l)}$. In each of these case we see that $v_Ts_1 = \pm v_T$. Thus, for simplicity we write $v_Ts_1 = (-1)^a v_T$ for some integer a. Also, depending on whether the box containing 1 is in the partition $\lambda^{(1)}$ or $\lambda^{(2)}$, we have $v_Te_1 = \pm v_T$. Again for simplicity, we write $v_Te_i = (-1)^b v_T$ for some integer b. Then

$$\begin{aligned} v_Ts_1e_1s_1e_1 &= (-1)^a v_Te_1s_1e_1 = (-1)^a(-1)^b v_Ts_1e_1 = (-1)^b v_Te_1 = v_T \\ &= (-1)^a v_Ts_1 = (-1)^b(-1)^a v_Te_1s_1 = (-1)^b v_Ts_1e_1s_1 = v_Te_1s_1e_1s_1 \ , \end{aligned}$$

as needed.

5. The relation $s_is_{i+1}s_i = s_{i+1}s_is_{i+1}$

The last relation from the list in (1) we wish to establish is $v_Ts_is_{i+1}s_i = v_Ts_{i+1}s_is_{i+1}$, for $1 \leq i \leq n-2$. Here we have the following cases regarding the relative positions of boxes containing i and $i+1$ and $i+2$:

(a) the three boxes are in one column or in one row;

(b) only two of the boxes are adjacent;

(c) none of the boxes is adjacent to another.

In case (a), note that v_Ts_i and v_Ts_{i+1} are not standard. Hence, $C_T(i) = C_T(i+1) = \pm 1$, depending on whether the boxes are lined up in a column or in a row. Thus, for simplicity, we write $C_T(i) = (-1)^a$ for some integer a. Then,

$$\begin{aligned} v_Ts_is_{i+1}s_i &= (-1)^a v_Ts_{i+1}s_i = (-1)^{2a} v_Ts_i = (-1)^{3a} = (-1)^{2a} v_Ts_{i+1} = (-1)^a v_Ts_is_{i+1} \\ &= v_Ts_{i+1}s_is_{i+1} \ . \end{aligned}$$

In case (b), consider first the case when the boxes contained i and $i+1$ are adjacent, and again write $C_T(i) = (-1)^a$ for some integer a. Then $U = \mathbb{C}\text{-span}\{v_T, v_T s_{i+1}, v_T s_{i+1} s_i\} \subseteq V^\lambda$ contains all the different scenarios satisfying the condition in case (b). Furthermore, U is invariant under $H = \langle s_i, s_{i+1}\rangle$, so it suffices to check the actions of H on U. Direct computation shows that we have the representation $\rho : H \to \mathrm{GL}_3(\mathbb{C})$ given by

$$s_i \mapsto \begin{pmatrix} (-1)^a & 0 & 0 \\ 0 & C_{Ts_{i+1}}(i) & 1 + C_{Ts_{i+1}}(i) \\ 0 & 1 - C_{Ts_{i+1}}(i) & -C_{Ts_{i+1}}(i) \end{pmatrix} \quad s_{i+1} \mapsto \begin{pmatrix} C_T(i) & 1 + C_T(i) & 0 \\ 1 - C_T(i) & -C_T(i) & 0 \\ 0 & 0 & (-1)^a \end{pmatrix}.$$

It follows that $v_T s_i s_{i+1} s_i = v_T s_{i+1} s_i s_{i+1}$.

We prove case (c) similarly. Let $U = \mathbb{C}\text{-span}\{v_T, v_{Ts_i}, v_{Ts_{i+1}}, v_{Ts_i s_{i+1}}, v_{Ts_i s_{i+1}}, v_{Ts_i s_{i+1} s_i}\}$ and H as above. Then U is invariant under H. The representation we seek is $\rho : H \to \mathrm{GL}_6(\mathbb{C})$ given by

$$s_i \mapsto \begin{pmatrix} C_T(i) & 1 + C_T(i) & 0 & 0 & 0 & 0 \\ 1 - C_T(i) & -C_T(i) & 0 & 0 & 0 & 0 \\ 0 & 0 & C_{Ts_{i+1}}(i) & 0 & 1 + C_{Ts_{i+1}}(i) & 0 \\ 0 & 0 & 0 & C_{Ts_i s_{i+1}}(i) & 0 & 1 + C_{Ts_i s_{i+1}}(i) \\ 0 & 0 & 1 - C_{Ts_{i+1}}(i) & 0 & -C_{Ts_{i+1}}(i) & 0 \\ 0 & 0 & 0 & 1 - C_{Ts_i s_{i+1}}(i) & 0 & -C_{Ts_i s_{i+1}}(i) \end{pmatrix}$$

$$s_{i+1} \mapsto \begin{pmatrix} C_{Ts_i s_{i+1}}(i) & 0 & 1 + C_{Ts_i s_{i+1}}(i) & 0 & 0 & 0 \\ 0 & C_{Ts_{i+1}}(i) & 0 & 1 + C_{Ts_{i+1}}(i) & 0 & 0 \\ 1 - C_{Ts_{i+1}}(i) & 0 & -C_{Ts_{i+1}}(i) & 0 & 0 & 0 \\ 0 & 1 - C_{Ts_{i+1}}(i) & 0 & -C_{Ts_{i+1}}(i) & 0 & 0 \\ 0 & 0 & 0 & 0 & C_T(i) & 1 + C_T(i) \\ 0 & 0 & 0 & 0 & 1 - C_T(i) & -C_T(i) \end{pmatrix}.$$

And again it follows that $v_T s_i s_{i+1} s_i = v_T s_{i+1} s_i s_{i+1}$ to complete the proof of the lemma. □

The next lemma we prove will assist us in proving item (ii) described in the outline above. We first define a generalization to the Murphy elements, which we will denote by $\mu_1, \ldots, \mu_n$. These elements are members of the group algebra $\mathbb{C}G(2,1,n)$. We will show that their actions on a standard tableau T of shape $\lambda = (\lambda^{(1)}, \lambda^{(2)})$ completely determines the filling of T.

Definition 7. For $k : 1, \ldots, n$, define the *generalized Murphy element* $\mu_k \in \mathbb{C}G(2,1,n)$ to be $\mu_k = ke_k + \frac{1}{2}\sum_{j=1}^{k-1}(1 + e_j e_k)(j,k)$, where $(j,k) \in S_n$ is a transposition, and the sum is taken to be zero if $k = 1$.

Lemma 8. *Let $\lambda = (\lambda^{(1)}, \lambda^{(2)})$ be an order pair of partitions with n total boxes, and consider the $G(2,1,n)$-module V^λ as defined above. Then for any $v_T \in V^\lambda$, where T is a standard tableau of* *shape λ, and for any generalized Murphy element μ_k, we have*

$$v_T \mu_k = D_T(k) v_T , \tag{2}$$

where we define $D_T(k)$ as follows:

$$D_T(k) = \begin{cases} (C(T(k)) + k) & \text{if the box containing } k \text{ is in } \lambda^{(1)} \\ (C(T(k)) - k) & \text{otherwise} \end{cases} .$$

Furthermore, the sequence $D_T(1), \ldots, D_T(n)$ uniquely determines the tableau T.

Proof. Note that for $i : 1, \ldots, k-2$ we have $s_{k-1}(1 + e_i e_{k-1})(i, k-1)s_{k-1} = (1 + e_i e_k)(i, k)$ and $s_{k-1}(k-1)e_{k-1}s_{k-1} = (k-1)e_k$. Thus,

$$\mu_k = s_{k-1}\mu_{k-1}s_{k-1} + \tfrac{1}{2}(1 + e_{k-1}e_k)s_{k-1} + e_k .$$

We prove the first part of the lemma by induction. By definition, $\mu_1 = e_1$ so $v_T \mu_1 = v_T$ if the box containing 1 is in $\lambda^{(1)}$ and $v_T \mu_1 = -v_T$ otherwise. Since we must have $C(T(1)) = 0$, the base case is established. In the following, suppose that the lemma is proven for μ_{k-1}.

Suppose that the boxes containing k and $k-1$ are in different partitions $\lambda^{(1)}$ and $\lambda^{(2)}$, not necessarily respectively. Then $v_T \left[\frac{1}{2}(1 + e_{k-1}e_k)\right] = \frac{1}{2}(v_T - v_T) = 0$, and also, by Remark 5, we know that $v_{Ts_{k-1}}$ is standard. If the box containing k is in the partition $\lambda^{(1)}$, then

$$\begin{aligned} v_T \mu_k &= v_T \left[s_{k-1}\mu_{k-1}s_{k-1} + \tfrac{1}{2}(1 + e_{k-1}e_k)s_{k-1} + e_k\right] = v_T \left(s_{k-1}\mu_{k-1}s_{k-1} + e_k\right) \\ &= v_{Ts_{k-1}}\mu_{k-1}s_{k-1} + v_T = (C(T(k)) + k - 1)v_{Ts_{k-1}}s_{k-1} + v_T = (C(T(k)) + k)v_T . \end{aligned}$$

Otherwise, if k is in the partition $\lambda^{(2)}$, then

$$\begin{aligned} v_T \mu_k &= v_T \left[s_{k-1}\mu_{k-1}s_{k-1} + \tfrac{1}{2}(1 + e_{k-1}e_k)s_{k-1} + e_k\right] = v_T \left(s_{k-1}\mu_{k-1}s_{k-1} + e_k\right) \\ &= v_{Ts_{k-1}}\mu_{k-1}s_{k-1} - v_T = (C(T(k)) - k + 1)v_{Ts_{k-1}}s_{k-1} - v_T = (C(T(k)) - k)v_T , \end{aligned}$$

as needed.

Now suppose that boxes containing k and $k-1$ are in the same partition $\lambda^{(l)}$. Then we have $v_T \left[\frac{1}{2}(1 + e_{k-1}e_k)\right] = \frac{1}{2}(v_T + v_T) = v_T$. Suppose first that $l = 1$. Then,

$$\begin{aligned} v_T \mu_k &= v_T \left(s_{k-1}\mu_{k-1}s_{k-1} + s_{k-1} + e_k\right) = v_T \left[\left(s_{k-1}\mu_{k-1} + 1\right)s_{k-1} + e_k\right] \\ &= \left(\tfrac{1}{C(T(k)) - C(T(k-1))} v_T \mu_{k-1} + \left(1 + \tfrac{1}{C(T(k)) - C(T(k-1))}\right) v_{Ts_{k-1}}\mu_{k-1} + v_T\right) s_{k-1} + v_T \\ &= \left(\tfrac{C(T(k-1)) + k - 1}{C(T(k)) - C(T(k-1))} v_T + \left(C(T(k)) + k - 1\right)\left(1 + \tfrac{1}{C(T(k)) - C(T(k-1))}\right) v_{Ts_{k-1}} + v_T\right) s_{k-1} + v_T \\ &= \left(\tfrac{C(T(k)) + k - 1}{C(T(k)) - C(T(k-1))} v_T + \left(C(T(k)) + k - 1\right)\left(1 + \tfrac{1}{C(T(k)) - C(T(k-1))}\right) v_{Ts_{k-1}}\right) s_{k-1} + v_T \\ &= (C(T(k)) + k - 1)v_T s_{k-1}s_{k-1} + v_T = (C(T(k)) + k)v_T . \end{aligned}$$

If, on the other hand, $l = 2$, then

$$\begin{aligned}
v_T\mu_k &= v_T\left(s_{k-1}\mu_{k-1}s_{k-1} + s_{k-1} + e_k\right) = v_T\left[\left(s_{k-1}\mu_{k-1} + 1\right)s_{k-1} + e_k\right] \\
&= \left(\tfrac{1}{C(T(k))-C(T(k-1))}v_T\mu_{k-1} + \left(1 + \tfrac{1}{C(T(k))-C(T(k-1))}\right)v_{Ts_{k-1}}\mu_{k-1} + v_T\right)s_{k-1} - v_T \\
&= \left(\tfrac{C(T(k-1))-k+1}{C(T(k))-C(T(k-1))}v_T + \left(C(T(k)) - k + 1\right)\left(1 + \tfrac{1}{C(T(k))-C(T(k-1))}\right)v_{Ts_{k-1}} + v_T\right)s_{k-1} - v_T \\
&= \left(\tfrac{C(T(k))-k+1}{C(T(k))-C(T(k-1))}v_T + \left(C(T(k)) - k + 1\right)\left(1 + \tfrac{1}{C(T(k))-C(T(k-1))}\right)v_{Ts_{k-1}}\right)s_{k-1} - v_T \\
&= (C(T(k)) - k + 1)v_Ts_{k-1}s_{k-1} - v_T = (C(T(k)) - k)v_T\ ,
\end{aligned}$$

to complete the proof by induction of the first part of the lemma.

Consider any $1 \le k \le n$. The minimal value $C(T(k))$ can attain is $-k+1$, which is possible only if all the numbers $1, \dots, k$ are placed in boxes of the same column in either $\lambda^{(1)}$ or $\lambda^{(2)}$. Similarly, the maximal value $C(T(k))$ can attain is $k - 1$, which is possible only if all the numbers $1, \dots, k$ are placed in boxes of the same row in either $\lambda^{(1)}$ or $\lambda^{(2)}$. Thus, $D_T(k)$ is bounded between 1 and $2k - 1$ if k is placed in a box in $\lambda^{(1)}$ and is bounded between $-2k + 1$ and -1 otherwise. That is, knowing the value of $D_T(k)$ determines in which partition k is placed. However, since the tableau T is standard, in any given partition $\lambda^{(l)}$ we know that no two different possible placing of the box containing k can have the same content $C(T(k))$. The second part of the lemma follows. □

While the previous lemma showed that the basis elements v_T are eigenvectors of the generalized Murphy elements μ_k, the next lemma shows that, up to scaling, they are the only such eigenvectors. The two lemmas together will be crucial in proving that two modules V^λ and V^γ are isomorphic only if $\lambda = \gamma$.

Lemma 9. *Let $\lambda = (\lambda^{(1)}, \lambda^{(2)})$ be an order pair of partitions with n total boxes, and consider the $G(2, 1, n)$-module V^λ as defined above. If $v \in V^\lambda$ is an eigenvector for all generalized Murphy elements $\mu_1, \dots, \mu_n$ then $v = cv_T$ for some standard tableau T of shape λ and complex number c.*

Proof. Let $v = \sum_T a_Tv_T$, where we sum over the standard tableaux of shape λ and the coefficients are from the complex field. By hypothesis, we have $v\mu_k = \alpha_kv$ for a complex number α_k. On the other hand, from equation (2) we know that $v\mu_k = \sum_T a_TD(T(k))v_T$. Thus, all but one a_T must vanish, and $c = \alpha_k = D(T(k))$. □

The last lemma we provide shows that standard tableaux form a sequence of tableaux differing by a single transposition. We will later construct a submodule of V^λ and show that it contains one of the basis elements v_T. The following lemma will assist us in showing that, in fact, all the basis elements are contained in the submodule, proving that the submodule is the entire module V^λ. We precede the lemma with an extension to the definition of a *column reading tableau* to the case of an ordered pair of partitions.

Definition 10. Let $\lambda = (\lambda^{(1)}, \lambda^{(2)})$ be a partition of n total boxes. Then the *column reading tableau of shape* λ is a tableau C of shape λ satisfying the following conditions: (a) its $\lambda^{(1)}$-filling is the same as the column reading tableau of shape $\lambda^{(1)}$; and (b) its $\lambda^{(2)}$-filling is the same as the column reading tableau of shape $\lambda^{(2)}$, but with each entry k replaced by $k+|\lambda^{(1)}|$.

Lemma 11. *If P and Q are standard tableaux of shape $\lambda=(\lambda^{(1)},\lambda^{(2)})$, then there exists a sequence of standard tableaux $P, P_{s_{i_1}}, P_{s_{i_1},s_{i_2}}, \dots, P_{s_{i_1},s_{i_2},\dots,s_{i_r}} = Q$.*

Proof. Consider the tableau P. Let i_1 be the maximal number such that either (a) the box containing i is strictly above and strictly to the right of the box containing i_1+1; or (b) the box containing i is in $\lambda^{(2)}$, while the box containing i_1+1 is in $\lambda^{(1)}$. If no such i_1 exists, then $P=C$, the column reading tableau of λ.

Otherwise, note the $P_{s_{i_1}}$ is a standard tableau as well. Again find a maximal number with respect to $P_{s_{i_1}}$ satisfying the conditions above and denote by i_2. Continue in this manner to generate i_j. The column reading tableau is characterized by the conditions above, and by Lemma 8 it is the only tableau satisfying these conditions. Furthermore, by choosing the maximal element to satisfy the conditions above, we guarantee that at no point will we arrive at a standard tableau we have seen already. Thus, the algorithm must terminate for some $P_{s_{i_1},s_{i_2},\dots,s_{i_s}} = C$.

Apply the same algorithm to Q to find $Q_{s_{j_1},s_{j_2},\dots,s_{j_t}} = C$. Then we have found a sequence $P, P_{s_{i_1}}, \dots, P_{s_{i_1},s_{i_2},\dots,s_{i_r}} = C = Q_{s_{j_1},s_{j_2},\dots,s_{j_t}}, \dots, Q_{s_{j_1}}, Q$ of standard tableaux, as needed. □

We are now ready to prove the classification theorem.

Proof (of the classification theorem, Theorem 4). As mentioned above, Lemma 6 proves item (i) of the outline: the V^λ are $G(2,1,n)$-modules.

Consider any two partitions $\lambda=(\lambda^{(1)},\lambda^{(2)})$ and $\gamma=(\gamma^{(1)},\gamma^{(2)})$ with n total boxes. Then we need to show that V^λ and V^γ are isomorphic as $G(2,1,n)$-modules if and only if $\gamma=\lambda$; this is item (ii) of the outline. The right-to-left implication is trivial. Conversely, suppose $\theta : V^\lambda \to V^\gamma$ is an isomorphism of $G(2,1,n)$-modules. Consider a standard tableau T of shape λ. Since θ is a homomorphism, we have $C(T(k))\big(v_T\big)\theta = \big(C(T(k))v_T\big)\theta = (v_T\mu_k)\theta = \big((v_T)\theta\big)\mu_k$. Hence, by Lemma 9 we know that $(v_T)\theta = cv_Q$ for a standard tableau Q and some complex number c. By Lemma 8 we know that the sequence $D(T(1)),\dots,D(T(n))$ and $D(Q(1)),\dots,D(Q(n))$ uniquely determine T and Q, which by the isomorphism θ we conclude to be the same. This completes the proof of (ii).

We have established that the V^λ defined in the statement of the theorem are non-isomorphic $G(2,1,n)$-modules. Next we prove item (iii): that each such V^λ is irreducible. Fix some nonzero $v_0=\sum_T a_T v_T \in V^\lambda$, and let $M=\{v_0 x \mid x\in \mathbb{C}G(2,1,n)\}$, a submodule of V^λ. We show that $M=V$, which in turn implies that the only nonzero submodule of V^λ is itself, or equivalently, that V^λ is irreducible.

For each standard tableau T, consider the element

$$\pi_T = \prod_{\substack{-2n\le j\le 2n\\ j\ne D_T(1)}} \frac{\mu_1-j}{D_T(1)-j} \prod_{\substack{-2n\le j\le 2n\\ j\ne D_T(2)}} \frac{\mu_2-j}{D_T(2)-j} \cdots \prod_{\substack{-2n\le j\le 2n\\ j\ne D_T(n)}} \frac{\mu_n-j}{D_T(n)-j}\,.$$

If we let π_T act on a basis element v_Q for $Q \neq T$, then the product vanishes due to j assuming a value $D(Q(k))$ for a k on which Q and T differ. On the other hand, if we let π_T act on v_T, then the ratios in the product are all 1, so π_T fixes v_T. That is, we have proved

$$v_Q \pi_T = \begin{cases} 0 & \text{if } Q \neq T \\ v_Q & \text{otherwise} \end{cases} . \tag{3}$$

Consider again $v_0 = \sum_T a_T v_T \in V^\lambda$. From equation (3), for each nonzero a_T we can write $v_T = v_0 \left(\frac{\pi_T}{a_T}\right) \in M$. By construction, there exists such a nonzero a_Q, as otherwise $v_0 = 0$. In particular, we have shown that at least one basis element v_Q is a member of M.

Suppose Qs_i is a standard tableau for some s_i. If the boxes containing i and $i+1$ are in different partitions $\lambda^{(1)}$ and $\lambda^{(2)}$, then $v_Q s_i = v_{Qs_i} \in M$. Otherwise, the boxes containing i and $i+1$ are in the same partitions $\lambda^{(l)}$. Then $v_Q s_i = \frac{1}{C(Q(i+1))-C(Q(i))} v_Q + \left(1 + \frac{1}{C(Q(i+1))-C(Q(i))}\right) v_{Qs_i}$. Note that it is impossible that $C(Q(i+1)) - C(Q(i)) = -1$, as this would imply that the two boxes are adjacent, contradicting the hypothesis that Qs_i is a standard tableau. It follows that $v_{Qs_i} \in M$. By Lemma 11 and induction on the number of transpositions, we have that $v_T \in M$ for all standard tableaux T. But then $M = V^\lambda$, so V^λ is irreducible to prove (iii).

To prove the last item (iv), we use an indirect counting argument. Let G be any finite group. In [2], Theorem 15.3, it is proven that the number of irreducible G-modules is equal to the number of conjugacy classes of G. In [1], it is shown that the conjugacy classes of $G(2,1,n)$ are indexed by ordered pairs of partitions $(\lambda^{(1)}, \lambda^{(2)})$ of n total boxes. Thus, there are no other irreducible $G(2,1,n)$-modules than the ones defined in the theorem. This concludes the proof of the theorem.

□

Using Theorem 4, it is possible to derive the character table of different groups $G(2,1,n)$ for an $n > 2$. We analyze the group of $G(2,1,3)$. We choose the following representatives for the conjugacy classes of $G(2,1,3)$:

$$\begin{array}{lllll} g_1 = 1 & g_2 = e_1e_2e_3 & g_3 = e_1e_2 & g_4 = e_1 & g_5 = s_1 \\ g_6 = e_1s_1 & g_7 = e_3s_1 & g_8 = e_1e_3s_1 & g_9 = s_1s_2 & g_{10} = e_1s_1s_2 \,. \end{array}$$

The corresponding sizes of the centralizers $|C_G(g_r)|$ are as follows:

$$\begin{array}{lllll} |C_G(g_1)| = 48 & |C_G(g_2)| = 48 & |C_G(g_3)| = 16 & |C_G(g_4)| = 16 & |C_G(g_5)| = 8 \\ |C_G(g_6)| = 8 & |C_G(g_7)| = 8 & |C_G(g_8)| = 8 & |C_G(g_9)| = 6 & |C_G(g_{10})| = 6 \,. \end{array}$$

The character table of $G(2,1,3)$ is the following:

	g_1	g_2	g_3	g_4	g_5	g_6	g_7	g_8	g_9	g_{10}
$\lvert C_G(g_r)\rvert$	48	48	16	16	8	8	8	8	6	6
χ_1	1	1	1	1	1	1	1	1	1	1
χ_2	1	1	1	1	−1	−1	−1	−1	1	1
χ_3	1	−1	1	−1	1	−1	−1	1	1	−1
χ_4	1	−1	1	−1	−1	1	1	−1	1	−1
χ_5	2	2	2	2	0	0	0	0	−1	−1
χ_6	2	−2	2	−2	0	0	0	0	−1	1
χ_7	3	−3	−1	1	1	1	−1	−1	0	0
χ_8	3	−3	−1	1	−1	−1	1	1	0	0
χ_9	3	3	−1	−1	1	−1	1	−1	0	0
χ_{10}	3	3	−1	−1	−1	1	−1	1	0	0

We finish this discussion with a conjecture of the classification theorem of $G(p, 1, n) \cong C_p \wr S_n$, where $n, p \geq 2$. The term a *standard tableau of shape* λ and the notation $C(T(i))$ appearing in this conjecture are the natural generalizations of the terms defined in Definitions 2 and 3.

Conjecture 12. *Let $G(p, 1, n)$ be the monomial matrix group of $n \times n$ matrices with exactly one nonzero entry in every column and row, and the nonzero entries are in C_p, the cyclic group of order p.*

1. *The irreducible $G(2, 1, n)$-modules V^λ are indexed by p-tuples $\lambda = (\lambda^{(1)}, \ldots, \lambda^{(p)})$ of partitions with n total boxes.*

2. *Their dimensions are $\dim(V^\lambda) = n_\lambda$, where n_λ is the number of standard tableaux of shape λ.*

3. *If $V^\lambda = \mathbb{C}\text{-span}\{v_T \mid T$ is a standard tableau of shape $\lambda\}$, then*

$$v_T e_i = e^{\frac{2\pi i r}{p}} v_T \,,$$
$$v_T s_i = C_T(i) v_T + (1 + C_T(i)) v_{Ts_i} \,,$$

where (i) the box containing i is in the partition $\lambda^{(r)}$ for some $1 \leq r \leq p$; (ii) $v_{Ts_i} = 0$ if v_{Ts_i} is not a standard tableau; and (iii) $C_T(i) = \frac{1}{C(T(i+1)) - C(T(i))}$ if the boxes containing i and $i+1$ are in the same partition $\lambda^{(k)}$ and zero otherwise.

References

[1] Ben Galin, *Math 156: Homework 4*, Spring 2007, Stanford University.

[2] Gordon James and Martin Liebeck, *Representations and characters of groups*, 2nd ed., Cambridge University Press, Cambridge, UK, 2006.